Journal of Applied Mathematics and Mechanics 73 (2009) 8–15

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The dynamics of an omni-mobile vehicle☆

A.A. Zobova, Ya.V. Tatarinov

Moscow, Russia

ARTICLE INFO

Article history:
Received 15 January 2008

ABSTRACT

The motion of an omni-mobile vehicle on a horizontal plane is considered. The wheels are modelled by absolutely rigid discs. Slippage in a certain direction, which makes a constant non- zero angle with the plane of a disc, is possible at the contact point of a wheel and the plane, and the planes of the discs are fixed with respect to the platform of the vehicle. The dynamic equations of motion are obtained for vehicles of this type with an arbitrary number and arrangement of the wheels. A complete qualitative description of the inertial motion of a vehicle is given (there are no control actions and it is assumed that there is no friction in the axes). The result is presented in the form of a phase portrait of the system. The motion of a vehicle is then considered in the case when control moments are applied to the axes of the wheels. The stability and branching of a certain class of steady motions of the vehicle are investigated. The domain of parameters is separated out where Andronov-Hopf bifurcation occurs with the formation of unstable limit cycles.

Omni-wheels were developed to construct manœuvrable means of transport (such as, for example, mobile robots and wheelchairs). Several rollers are fixed onto the periphery of the disc of such a wheel, so that only one of the rollers is able to contact the supporting surface and this roller can freely rotate about a certain fixed axis in the disc of the wheel. The axis of rotation of the roller is directed either along the tangent of the periphery of the disc (an omni-wheel) or it is turned around the radius of the disc at an angle of 45° (a mecanum-wheel). This construction enables a wheel, supported on a roller and maintaining the orientation of its plane, to move easily along a straight line at a fixed non-zero angle to the plane of the wheel. A practical approach to the design of vehicles with such wheels has been thoroughly developed by foreign authors (see, for example, Refs 1 and 2 and the bibliography in these papers). The motion (both free and controlled) of a vehicle with roller-carrying wheels of a certain specific construction has been studied in Ref. 3. The equations of the free and controlled motions of an arbitrarily configured vehicle on a horizontal plane are obtained and analysed below using the methods of analytical mechanics and stability theory. A concise method of obtaining the equations of motion for mechanical systems with non-holonomic constraints, proposed earlier in Refs 4 and 5, is also described in detail.

1. Formulation of the problem

Consider the motion of an omni-mobile vehicle of mass M with N roller-carrying wheels of radius R over a rough horizontal plane OXY (further, we assume that the dimensionally independent quantities M and R are equal to unity). The planes of the wheels are vertical and they are fixed with respect to the platform of the vehicle. Suppose Q is a certain fixed point of the vehicle, S is the centre of mass of the vehicle and $QS = \Delta$. We now introduce the vectors $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ in the following manner (Fig. 1): $\mathbf{e}_\eta = \overrightarrow{QS}/|QS|$, $\mathbf{e}_\xi \perp \mathbf{e}_\eta$, $|\mathbf{e}_\xi| = |\mathbf{e}_\eta| = 1$ (without loss of generality, we will assume that the vectors $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$ are horizontal). The angle between the vector $\mathbf{e}_\xi$ and the OX axis is denoted by θ. The position of the vehicle is completely determined by the x and y coordinates of the centre of mass in the OXY plane, the angle θ and the angle χ_i ($i = 1, \ldots, N$) of the proper rotation of the wheels.

The following constraints are imposed on the system: the direction of the velocity of the lowest point of each disc, which simulates a roller-carrying wheel, is at a certain constant angle to the plane of the wheel (this velocity is actually directed perpendicularly to the axis of the roller on which the wheel leans at a given instant). In order to obtain the equations of these constraints, we will introduce the following notation (see Fig. 1). Suppose $\mathbf{e}_i$ is a unit vector which leaves from the point Q and is directed into the centre of the i-th wheel P_i (henceforth

☆ *Prikl. Mat. Mekh.* Vol. 73, No. 1, pp. 13–22, 2009.

E-mail address: azobova@mail.ru (A.A. Zobova).

0021-8928/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2009.03.013

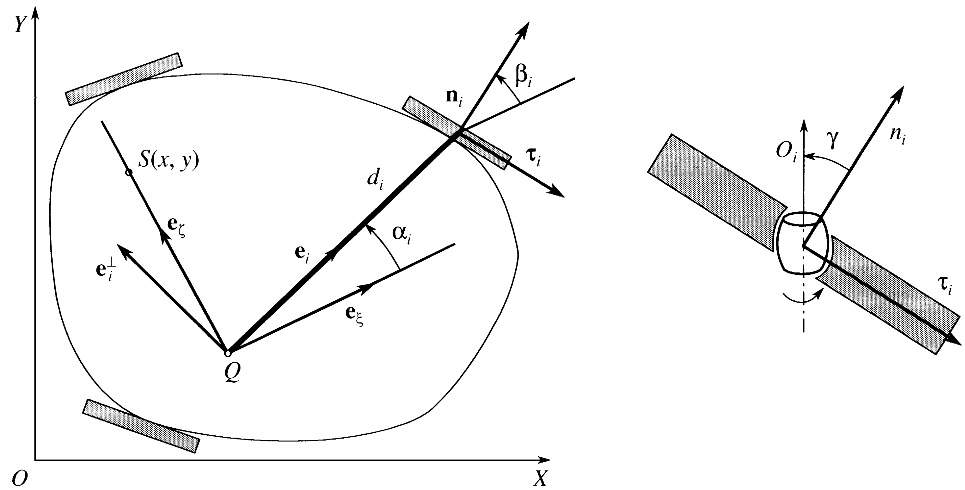

Fig. 1.

$i \in \{1, \ldots, N\}$, $k \in \{1, \ldots, 3+N\}$, $s, s' \in \{1, 2, 3\}$ and repeated indices under a summation sign run through the above mentioned sets), and $\mathbf{e}_i^\perp$ is a vector which is perpendicular to it. We will denote the angle between the vectors $\mathbf{e}_\xi$ and $\mathbf{e}_i$ by α_i and the angle between the $Q\xi$ axis and the perpendicular $\mathbf{n}_i$ to the plane of the wheel by β_i. Suppose δ_i is the distance between the points Q and P_i and τ_i is a horizontal vector which is tangential to the plane of the wheel. Suppose $\mathbf{o}_i$ is a vector corresponding to the direction of the axis of a roller and γ_i is the angle between the vectors $\mathbf{o}_i$ and $\mathbf{n}_i$. For omni-wheels, the angle γ_i is equal to $\pi/2$ and, for mecanum-wheels, $\gamma_i = \pi/4$.

We now introduce the pseudovelocities ν_1 and ν_2 as the projections of the velocity of the centre of mass on the vectors $\mathbf{e}_\xi$ and $\mathbf{e}_\eta$. Then,

$$\dot{x} = \cos\theta\nu_1 - \sin\theta\nu_2, \quad \dot{y} = \sin\theta\nu_1 + \cos\theta\nu_2 \tag{1.1}$$

The velocity $\boldsymbol{\upsilon}_i$ of the lowest point of the i-th wheel

$$\boldsymbol{\upsilon}_i = (\nu_1 + \Delta\dot{\theta})\mathbf{e}_\xi + \nu_2\mathbf{e}_\eta + \dot{\theta}\delta_i\mathbf{e}_i^\perp + \dot{\chi}_i\boldsymbol{\tau}_i \tag{1.2}$$

is perpendicular to the vector $\mathbf{o}_i$. Hence, the equations of the constraints have the form

$$\dot{\chi}_i\sin\gamma_i = \cos(\beta_i + \gamma_i)\nu_1 + \sin(\beta_i + \gamma_i)\nu_2 + (\Delta\cos(\beta_i + \gamma_i) + \delta_i\sin(\beta_i + \gamma_i - \alpha_i))\dot{\theta}$$

2. The equations of inertial motion and their analysis

We will now consider the inertial motion of the vehicle, that is, we will assume that, apart from the ideal reactions of the constraints, only a gravitational force acts on the system. In order to derive the equations of motion, we will make use of the concise form proposed earlier.[4,5] To do this, we will first write out the expression for the kinetic energy

$$2T = \dot{x}^2 + \dot{y}^2 + \Lambda^2\dot{\theta}^2 + \lambda^2(\dot{\chi}_1^2 + \dot{\chi}_2^2 + \ldots + \dot{\chi}_n^2)$$

Here, Λ^2 is the total moment of inertia with respect to the vertical axis passing through the point S and λ^2 is the moment of inertia of each wheel with respect to the $\mathbf{n}_i$ axis (they are assumed to be identical). We now introduce the pseudovelocity

$$\nu_3 = \Lambda\dot{\theta} \tag{2.1}$$

Thus, constraints which, in matrix form, can be written in the following manner

$$\|\dot{\chi}_1, \ldots, \dot{\chi}_N\|^T = \Xi\|\nu_1, \nu_2, \nu_3\|^T, \quad \Xi = \|\sigma_{is}\|$$
$$\sigma_{i1} = \frac{\cos(\beta_i + \gamma_i)}{\sin\gamma_i}, \quad \sigma_{i2} = \frac{\sin(\beta_i + \gamma_i)}{\sin\gamma_i}, \quad \sigma_{i3} = \frac{\Delta\cos(\beta_i + \gamma_i) + \delta_i\sin(\beta_i + \gamma_i - \alpha_i)}{\Lambda\sin\gamma_i} \tag{2.2}$$

are imposed on the mechanical system, the position of which is given by the coordinates $\{q_k\} \equiv \{x, y, \theta, \chi_1, \chi_2, \ldots, \chi_N\}$ with Lagrangian $L = T$ (the potential energy of the gravity force does not vary since the vehicle moves along a horizontal plane).

We denote the coefficients of the pseudovelocities ν_s in the sum $\Sigma p_k\dot{q}_k$ when account is taken of relations (1.1), (2.1) and (2.2) by P_s. They have the form

$$P_1 = p_x\cos\theta + p_y\sin\theta + \sum p_\chi^i\sigma_{i1}, \quad P_2 = -p_x\sin\theta + p_y\cos\theta + \sum p_\chi^i\sigma_{i2},$$
$$P_3 = \frac{p_\theta}{\Lambda} + \sum p_\chi^i\sigma_{i3}$$

Here, $\{p_k\} \equiv \{p_x, p_y, p_\theta, p_\chi^1, \ldots, p_\chi^N\}$ are the formal canonical momenta, that is, the variables which, in conjunction with q_k, enable us to calculate the Poisson bracket.

The equations of motion can be written in the form

$$\frac{d}{dt}\frac{\partial L^*}{\partial \nu_s} + \{P_s, L^*\} = \sum \nu_{s'}\{P_s, P_{s'}\}^*$$

Here, $L^* = \nu^T\mathbf{A}\nu/2$ is the Lagrangian after substitution of the constraints, $\nu^T = (\nu_1, \nu_2, \nu_3)$ is the three-dimensional pseudovelocity vector and the matrix $\mathbf{A} = \mathbf{E} + \lambda^2\ \Xi^T\Xi$.

Note that $\{P_s, L^*\} = 0$ and we then find

$$\{P_1, P_2\} = 0, \quad \{P_1, P_3\} = \frac{\partial P_1}{\partial \theta}\frac{\partial P_3}{\partial p_\theta} = \frac{-p_x\sin\theta + p_y\cos\theta}{\Lambda}$$

$$\{P_2, P_3\} = \frac{\partial P_2}{\partial \theta}\frac{\partial P_3}{\partial p_\theta} = \frac{-p_x\cos\theta - p_y\sin\theta}{\Lambda}$$

Now, instead of p_x, p_y, we substitute the generally known expressions $p_x = \partial T/\partial\dot{x} = \dot{x}$, $p_y = \partial T/\partial\dot{y} = \dot{y}$ and take relation (1.1) into account. We obtain that

$$\{P_1, P_3\}^* = \Lambda^{-1}\nu_2, \quad \{P_2, P_3\}^* = -\Lambda^{-1}\nu_1$$

Consequently, the dynamic equations take the form

$$\frac{d}{dt}\frac{\partial T^*}{\partial \nu_1} = \frac{\nu_2\nu_3}{\Lambda}, \quad \frac{d}{dt}\frac{\partial T^*}{\partial \nu_2} = -\frac{\nu_1\nu_3}{\Lambda}, \quad \frac{d}{dt}\frac{\partial T^*}{\partial \nu_3} = 0$$

and they can be written in matrix form as follows:

$$\mathbf{A}\dot{\nu} = \frac{\mathbf{a}}{\Lambda}, \quad \mathbf{a} = \|\nu_2\nu_3, -\nu_1\nu_3, 0\|^T \tag{2.3}$$

The system admits of an energy integral $T^* \equiv \text{const}$ and a linear integral $\partial T^*/\partial\nu_3 \equiv \text{const}$ and, also, an invariant measure $\mu = |\nu_3|^{-1}d\nu_1 \wedge d\nu_2 \wedge d\nu_3$. In the space of the pseudovelocities ν_1, ν_2, ν_3, the trajectories of the system lie in a section of the ellipsoid $\nu A \nu^T = 2h = \text{const}$, which is the level of the energy integral and the plane of the line integral, by $\partial T^*/\partial\nu_3 \equiv \text{const}$.

We will now consider a three-wheeled vehicle with the following geometry (Fig. 2):

$$\alpha_1 = 0, \quad \alpha_2 = \pi/2, \quad \alpha_3 = \pi, \quad \beta_i = \alpha_i, \quad \gamma_i = \pi/2, \quad i = 1, 2, 3$$

$$\delta_1/\Lambda = \delta_3/\Lambda = \sigma > 0, \quad (\delta_2 - \Delta)/\Lambda = \rho > 0$$

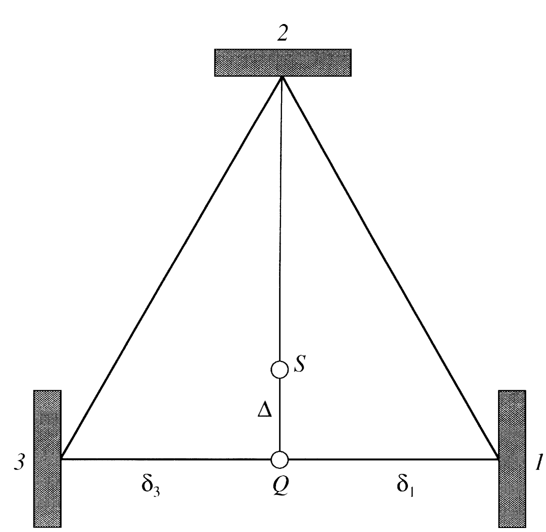

Fig. 2.

The matrices Ξ and $\mathbf{A}$ then have the form

$$\Xi^0 = \begin{Vmatrix} 0 & 1 & \sigma \\ -1 & 0 & \rho \\ 0 & -1 & \sigma \end{Vmatrix}, \quad \mathbf{A}^0 = \begin{Vmatrix} A_1 & 0 & -\kappa A_3 \\ 0 & A_2 & 0 \\ -\kappa A_3 & 0 & A_3 \end{Vmatrix}$$

$$A_1 = 1 + \lambda^2, \quad A_2 = 1 + 2\lambda^2, \quad A_3 = 1 + (2\sigma^2 + \rho^2)\lambda^2, \quad \kappa = \rho\lambda^2/A_3$$

Note that κ (apart from a positive factor) is the distance from the front wheel to the centre of mass of the system. In this case, the equations of motion take the form

$$A_1\dot{\nu}_1 - \kappa A_3\dot{\nu}_3 = \frac{\nu_2\nu_3}{\Lambda}, \quad A_2\dot{\nu}_2 = -\frac{\nu_1\nu_3}{\Lambda}, \quad -\kappa A_3\dot{\nu}_1 + A_3\dot{\nu}_3 = 0 \tag{2.4}$$

We will assume that $\kappa \neq 0$ and fix a certain level of the linear integral $-\kappa\nu_1 + \nu_3 = K = \text{const}$ of this system. The projections of the phase trajectories of system (2.4) lying in the plane of the linear integral are described in the (ν_3, ν_2) plane by the system of equations

$$(A_1 - \kappa^2 A_3)\dot{\nu}_3 = \frac{\kappa\nu_2\nu_3}{\Lambda}, \quad A_2\dot{\nu}_2 = -\frac{(\nu_3 - K)\nu_3}{\Lambda\kappa} \tag{2.5}$$

In the (ν_3, ν_2) plane, the trajectories of the system with a different energy h belong to concentric ellipses with their centre on the $\nu_2 = 0$ axis (the phase portrait is shown in Fig. 3, *a*). The centre of the ellipses $\nu_2 = 0$, $\nu_3 = -K$ corresponds to steady rotation of the vehicle about the vertical axis passing through the centre of mass ($\nu_1 \equiv 0$, $\nu_2 \equiv 0$, $\dot{\theta} \equiv \text{const}$, $x \equiv x_0$, $y \equiv y_0$). The special line $\nu_3 = 0$ corresponds to the steady rectilinear motion of the vehicle (the angle θ is constant during the motion and the axis of symmetry does not have to be parallel to the path). Analysis of Eqs (2.5) shows that, in the half-space $\nu_3 > 0$, motion along the ellipses occurs in a clockwise direction and, in the half-space $\nu_3 < 0$, in an anticlockwise direction. It then follows that the steady rectilinear motions ($\nu_2 = \text{const}$, $\nu_3 = 0$) are stable when $\nu_2 < 0$ and unstable when $\nu_2 > 0$ (here, we have in mind stability with respect to part of the variables, with respect to the pseudovelocities ν_s and the modulus of the velocity of the centre of mass $(\dot{x}^2 + \dot{y}^2)^{1/2}$). The physical meaning of this condition lies in the fact that a rectilinear motion is stable if and only if the centre of mass is located behind the axis of the parallel wheels. It is interesting to note that, in the case of the inertial motion of another model of a mobile vehicle with conventional wheels,[6] the condition for the stability of the steady rectilinear motions of the vehicle has exactly the opposite meaning.

The motion of the vehicle that corresponds to periodic motions of a representative point along the ellipses is as follows: the centre of mass describes a multipetal-shaped curve and, at the same time, the platform rotates about a vertical axis in a certain constant direction. The motions of the vehicle that correspond to aperiodic (asymptotic) motions (that is, high energy motions) is as follows: when $t \to \infty$, the motion of the vehicle tends asymptotically to steady rectilinear motion.

We will now consider the case when $\kappa = 0$. In this case, the matrix $\mathbf{A}$ is diagonal and any motion of the vehicle occurs at a constant angular velocity $\dot{\theta} = \omega = \text{const}$. The energy integral has the form $A_1\nu_1^2 + A_2\nu_2^2 = \text{const}$; the centre of mass describes a multipetal-shaped curve on the supporting plane and, after each period $2\pi\omega^{-1}\sqrt{A_1A_2}$, identical segments (petals) are passed over but they are rotated with respect to one another by a certain angle. In the case of the inertial motion of a vehicle of another construction[3] (the wheels carrying rollers are arranged at the vertices of a regular triangle and orientated perpendicular to the bisectrices of the corresponding angles), the matrix $\mathbf{A}$ is diagonal and, moreover, $A_1 = A_2$. It has been shown[3] that, in this case, the motion of the vehicle when $\omega \neq 0$ is a cylindrical precession: the centre of mass moves uniformly along a circle and the platform rotates uniformly around the vertical axis passing through the centre of mass; if $\omega = 0$, the vehicle moves uniformly and rectilinearly.

Note that, in system (2.3) with an arbitrary symmetry of the positive-definite matrix $A = ||a_{rs}||$, two types of steady motions also exist: uniform rotations $\nu_1 \equiv 0$, $\nu_2 \equiv 0$, $\nu_3 = \nu_3^0$ and uniform rectilinear motions $\nu_1 = \nu_1^0$, $\nu_1 = \nu_2^0$, $\nu_3 = 0$. Considering the stability of the rectilinear motions with respect to part of the variables, in the first approximation, we obtain the following result: the necessary condition for

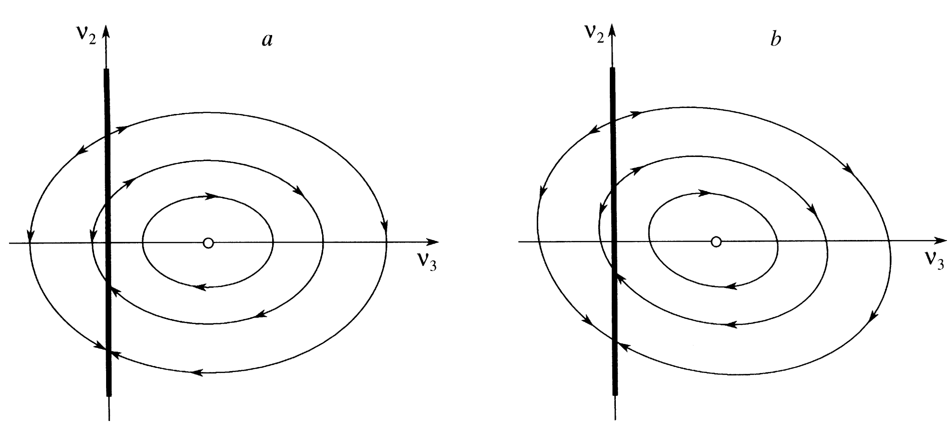

Fig. 3.

the rectilinear motion to be stable is the inequality

$$(a_{12}a_{23}-a_{13}a_{22})\nu_2^0+(a_{11}a_{23}-a_{13}a_{12})\nu_1^0>0$$

The projections of the phase trajectories onto the (ν_3, ν_2) plane in the case when the off-diagonal elements a_{12}, a_{23} in the matrix **A** are non-zero are shown in Fig. 3, *b*.

3. The equations of controlled motions

We will now consider the controlled motion of the vehicle. Suppose control moments $\mathbf{M}_i=(c_1U_i-c_2\dot{\chi}_i)\mathbf{n}_i$, where U_i are the control voltages, are applied to the wheels from the body. Such a description of drives with dc motors is generally accepted in the treatment of the dynamics of controlled vehicles (see Ref. 6).

Adding the generalized forces corresponding to the control moments to the right-hand sides of Eqs (2.3), we write the equations of the controlled motions

$$\mathbf{A}\dot{\boldsymbol{\nu}} = \Lambda^{-1}\mathbf{a}+c_1\boldsymbol{\Xi}^T\mathbf{U}-c_2\boldsymbol{\Xi}^T\boldsymbol{\Xi}\boldsymbol{\nu}, \quad \mathbf{U} = \|U_1, \ldots, U_N\|^T$$

These equations can be simplified by introducing the linear change with constant coefficients of the control parameters

$$\mathbf{w} = \|w_1, w_2, w_3\|^T = c_1c_2^{-2}\boldsymbol{\Xi}^T\mathbf{U}$$

and the dimensionless time $\tau=c_2t$. We obtain

$$\mathbf{A}\dot{\boldsymbol{\nu}} = \Lambda^{-1}\mathbf{a}+\mathbf{w}-\boldsymbol{\Xi}^T\boldsymbol{\Xi}\boldsymbol{\nu}$$

Here w_1, w_2, w_3 are the new control parameters and a derivative with respect to the new time is denoted by a dot.

For the vehicle shown in Fig. 2, the equations of the controlled motions take the form

$$A_1\dot{\nu}_1-\kappa A_3\dot{\nu}_3 = \frac{\nu_2\nu_3}{\Lambda}+w_1-(\nu_1-\rho\nu_3), \quad A_2\dot{\nu}_2 = -\frac{\nu_1\nu_3}{\Lambda}+w_2-2\nu_2$$

$$-\kappa A_3\dot{\nu}_1+A_3\dot{\nu}_3 = w_3-(-\rho\nu_1+\zeta^2\nu_3); \quad \zeta^2 = 2\sigma^2+\rho^2 \tag{3.1}$$

4. Steady controlled motions

Following the approach described earlier,[7] we consider the critical points of system (3.1)

$$\nu_1(t)\equiv p_1 = \text{const}, \quad \nu_2(t)\equiv p_2 = \text{const}, \quad \nu_3(t)\equiv \Lambda\omega = \text{const}$$

for constant controls. The steady motions (SM) of the vehicle in a plane correspond to critical points. The equations of the SM have the form

$$p_2\omega-p_1+\rho\Lambda\omega = -w_1, \quad p_1\omega+2p_2 = w_2, \quad \rho p_1-\zeta^2\Lambda\omega = -w_3 \tag{4.1}$$

For each fixed set **w**, from one to three steady motions of the vehicle exist.

In the neighbourhood of the SM $\nu_1=p_1+\delta p_1$, $\nu_2=p_2+\delta p_2$, $\nu_3=\Lambda(\omega+\delta\omega)$, the dynamic equations can be linearized:

$$A^0\left\|\begin{matrix}\delta\dot{p}_1\\ \delta\dot{p}_2\\ \delta\dot{\omega}\end{matrix}\right\| = \left\|\begin{matrix}-1 & \omega & p_2+\rho\Lambda\\ -\omega & -2 & -p_1\\ \Lambda^{-1}\rho & 0 & -(2\sigma^2+\rho^2)\end{matrix}\right\|\left\|\begin{matrix}\delta p_1\\ \delta p_2\\ \delta\omega\end{matrix}\right\| \tag{4.2}$$

In this equation, the values of p_1, p_2, ω must be expressed in terms of the parameters w_1, w_2, w_3 from Eqs (4.1).

We will now consider those values of the control parameters for which a rectilinear SM $\dot{\chi}_1=-\dot{\chi}_3$ is possible (the parallel wheels rotate uniformly in one direction). The control voltages must be connected by the relation $U_1=-U_3$, from which it follows that $w_3=-\rho w_1\equiv\text{const}$. Then, apart from the rectilinear SMs ($p_1=w_1$, $p_2=w_2/2$, $w=0$), rotational SMs also exist: the centre of mass moves along a circle in the support plane and, at the same time, the vehicle rotates uniformly around a vertical passing through the centre of mass. Here, the constants p_1, p_2, ω and w_1, w_2 must interrelated as follows:

$$\Lambda\zeta^2\omega^2+\rho w_1\omega+(4\Lambda\sigma^2-w_2\rho) = 0$$

$$p_1 = w_1+\rho^{-1}\zeta^2\omega, \quad p_2 = 2\rho^{-1}\Lambda\sigma^2$$

These families of motions respectively represent a straight line and a parabola in the bifurcation diagram (w_2, ω) (Fig. 4). The branches of the parabola are directed to the right and its vertex is located at the point

$$\omega_0 = -\frac{\rho w_1}{2(\rho^2+2\sigma^2)}, \quad (w_2)_0 = \frac{4\Lambda\sigma^2}{\rho}-\frac{\rho w_1^2}{2\Lambda(\rho^2+2\sigma^2)}$$

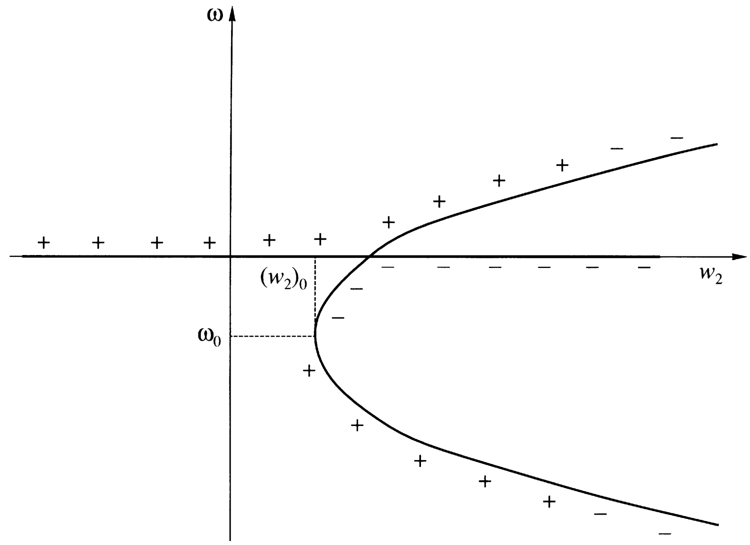

Fig. 4.

Analysis of the characteristic polynomial of Eqs (4.2) for the rectilinear motions gives the condition for the asymptotic stability of these motions $4\Lambda\sigma^2 - \rho w_2 > 0$. It follows from this condition that a motion for which the projection of the velocity of the centre of mass on the $Q\eta$ axis does not exceed a certain magnitude $\nu_2 \equiv p_2 = w_2/2 < 2\Lambda\sigma^2/\rho$ will be asymptotically stable (they are labelled with a plus sign). The remaining rectilinear motions are asymptotically unstable (they are labelled with a minus sign).

We will now investigate the stability of the rotational SMs. The characteristic equation of system (4.2) in their neighbourhood has the form

$$M_3\mu^3 + M_2\mu^2 + M_1\mu + M_0 = 0$$

$$M_3 > 0, \quad M_2 > 0, \quad M_1 > 0, \quad M_0 = (2(\rho^2 + 2\sigma^2)\omega^2 + \rho w_1\Lambda^{-1}\omega)$$

Hence, when $\omega = 0$ and $\omega = \omega_0$, the real root of the characteristic equation changes sign and, consequently, so does the nature of the stability. Analysis of the expression $\mathcal{R} = M_1M_2 - M_0M_3$ shows that values of the inertial and geometric parameters exist for which $\mathcal{R} < 0$ in a certain domain $(-\infty, \omega_-) \cup (\omega_+, +\infty)$ and, $\mathcal{R} \geq 0$ in the complement of the above mentioned domain (at the same time, $(\omega_-, \omega_+) \supset (\omega_0, 0)$).

We will now consider the type of the bifurcation at the points $\omega = \omega_\pm$ when $w_3 = -\rho w_1 = 0$ in greater detail. In this case, the critical points $\omega_\pm$ are solutions of the equation

$$\begin{aligned}
&\mathcal{R} = f(\lambda, \rho, \sigma)\omega_\pm^2 + g(\lambda, \rho, \sigma)\\
&f(\lambda, \rho, \sigma) = (32\sigma^4 + 16\sigma^2\rho^2)\lambda^6 + (40\sigma^4 + 8\rho^4 + 36\sigma^2\rho^2 + 24\sigma^2 + 8\rho^2)\lambda^4 +\\
&+ (8\sigma^4 + 2\rho^4 + 8\sigma^2\rho^2 + 22\sigma^2 + 10\rho^2 + 4)\lambda^2 - 2\sigma^2 - \rho^2 + 3\\
&g(\lambda, \rho, \sigma) = 32\sigma^4\lambda^6 + (72\sigma^4 + 32\sigma^2\rho^2 + 32\sigma^2)\lambda^4 +\\
&+ (48\sigma^4 + 8\rho^4 + 40\sigma^2\rho^2 + 48\sigma^2 + 16\rho^2 + 8)\lambda^2 + 8\sigma^4 + 2\rho^4 + 8\sigma^2\rho^2 + 16\sigma^2 + 8\rho^2 + 6
\end{aligned} \tag{4.3}$$

This equation has a solution for the values of the mass-inertia characteristics (λ, ρ, σ) that $f(\lambda, \rho, \sigma) < 0$. Analysis of the function $f(\lambda, \rho, \sigma)$ shows that the maximum dimensionless moment of inertia of a wheel λ_{max} for which slanting SMs can lose stability is less than 0.15.

When $w_3 = -\rho w_1 = 0$, the non-linear system of equations of a motion which has been perturbed in the neighbourhood of $\omega = \omega_\pm$, has the form

$$\mathbf{A}\left\|\begin{matrix} \delta\dot{p}_1 \\ \delta\dot{p}_2 \\ \Lambda\delta\dot{\omega} \end{matrix}\right\| = \left\|\begin{matrix} -1 & \omega_\pm & \rho^{-1}\zeta^2 \\ -\omega_\pm & -2 & -\rho^{-1}\zeta^2\omega_\pm \\ \rho & 0 & -\zeta^2 \end{matrix}\right\|\left\|\begin{matrix} \delta p_1 \\ \delta p_2 \\ \Lambda\delta\omega \end{matrix}\right\| + \Lambda^{-1}\left\|\begin{matrix} \delta p_2(\Lambda\delta\omega) \\ -\delta p_1(\Lambda\delta\omega) \\ 0 \end{matrix}\right\|$$

By the change of variables

$$\|\delta p_1, \delta p_2, \Lambda\delta\omega\|^T = \|\alpha_{ss'}\|\xi, \quad \boldsymbol{\xi} = \|\xi_1, \xi_2, \xi_3\|^T$$

this system can be reduced to the canonical form

$$\dot{\xi}_1 = -M_2\xi_1 + Q_1(\xi), \quad \dot{\xi}_2 = -\sqrt{M_1}\xi_3 + Q_2(\xi), \quad \dot{\xi}_3 = \sqrt{M_1}\xi_2 + Q_3(\xi)$$

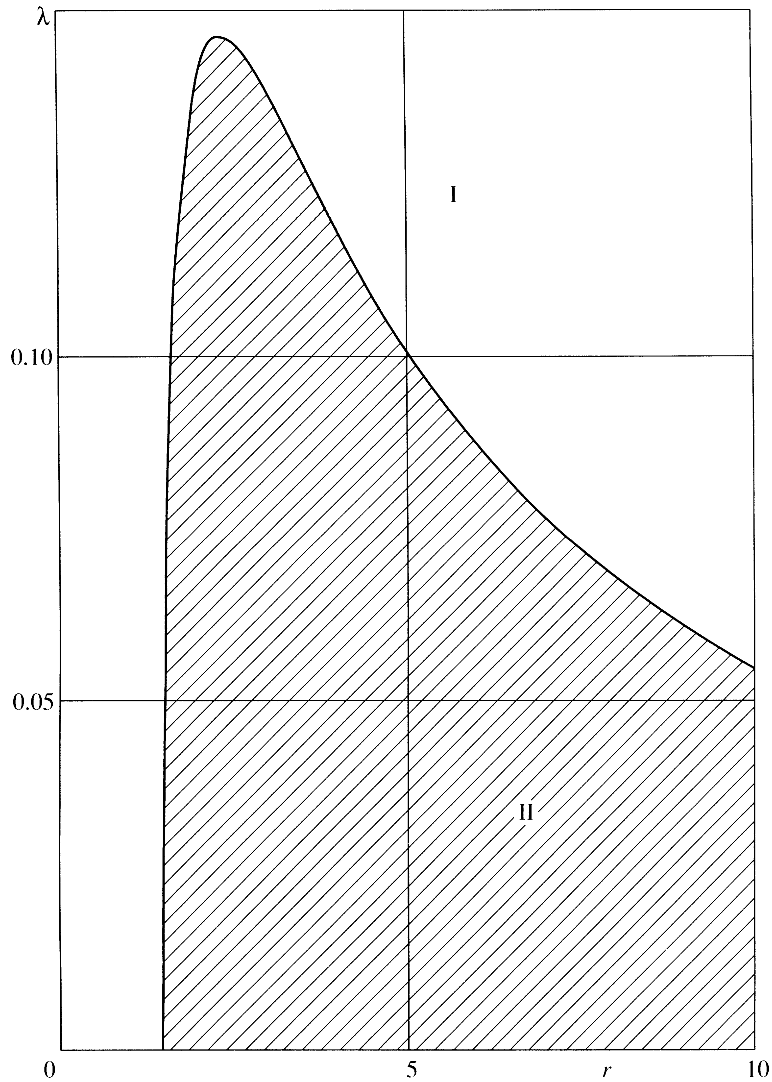

Fig. 5.

The coefficients of the change of variable $\alpha_{ss'}$ and the coefficients of the characteristic equation M_1 and M_2 are only constructed using the linear part of the perturbation equation (see Ref. 8) and therefore depend on $\omega_\pm$, λ^2, ρ^2, σ^2. Since $\omega_\pm$ is the solution of Eq. (4.3), $\alpha_{ss'}$, M_1, M_2 only depend on the parameters λ^2, ρ^2, σ^2 (and are independent of the parameter Λ). The parameter Λ occurs in the functions Q_s, which are quadratic in ξ_s, as follows:

$$Q_s(\boldsymbol{\xi}) = \Lambda^{-1} A_s^{s's''}(\lambda^2, \rho^2, \sigma^2)\xi_{s'}\xi_{s''}$$

The first Lyapunov coefficient L_1 is a homogeneous quadratic form of the coefficients of the functions Q_i and the sign of L_1 is therefore independent of the magnitude of Λ. Hence, we obtain that the qualitative bifurcation pattern when $\omega = \omega_\pm$ is independent of the magnitude of Λ.

So, when $w_3 = -\rho w_1 = 0$, it is possible to construct two domains in the (λ, ρ, σ) parameter space and there are qualitative differences in the behaviour of the system in these two domains. In domain I, the rotational SMs are always stable (critical values of $\omega_\pm$ do not exist) and, in domain II, the unstable limit cycles collapse to rotational SMs when w_2 is increased (the results of calculations show that $L_1 > 0$ in the whole of the domain $f(\lambda, \rho, \sigma) < 0$ and, for this $\mathcal{R}$, the sign changes from plus to minus on passing through the critical value of w_2). The characteristic form of the domains I and II in a section of the parameter space with the planes $\sigma = r\cos\varphi$, $\rho = r\sin\varphi$, $\varphi = \text{const}$ is shown in Fig. 5 (the plane $\varphi = \pi/4$ is shown and domain II is hatched).

Acknowledgements

We wish to thank A. V. Karapetyan for his interest in the research and his advice.

This research was supported by the Russian Foundation for Basic Research (07-01-00290, 06-08-01574) and the “Universities of Russia” Programme.

References

1. Williams II RL, Carter BE, Gallina P, Rosati G. Dynamic model with slip for wheeled omni-directional robots. *IEEE Trans Robotics and Automation* 2002;**18**(3):285–93.
2. Ashmore M, Barnes N. Omni-drive robot motion on curved paths: the fastest path between two points is not a straight line. In: *Proc. 15th Australian Joint Conf. on Artificial Intelligence*. 2002. p. 225–36.
3. Martynenko YuG, Formal'skii AM. The motion of a mobile robot with roller-carrying wheels. *Izv Ross Akad Nauk Theory and Control Systems* 2007;**6**:142–9.
4. Tatarinov YaV. The equations of classical mechanics in a new form. *Vestnik MGU Ser 1 Matematika Mekhanika* 2003;**3**:67–76.
5. Tatarinov YaV. *The Equations of Classical Mechanics in Concise Forms*. Moscow: Izd Tsentra Prikladnykh Issledovanii pri Mekhaniko-Matematicheskom Fakul'tete MGU (Publishing House of the Centre for Applied Research at the Mechanical-Mathematical Faculty of MGU);2005.
6. Budanov VM, Devyanin YeA. The motion of wheeled robots. *Prikl Mat Mekh* 2003;**67**(2):244–55.
7. Karapetyan AV, Salmina MA. Poincaré-Chetayev and Andronov-Hopf bifurcations in the dynamics of a three-wheeled robot. *Vestnik MGU Ser 1 Matematika Mekhanika* 2006;**2**:56–9.
8. Bautin NN. *The Behaviour of Dynamical Systems Close to the Boundaries of the Instability Domain*. Moscow: Nauka; 1984.

Translated by E.L.S.